Uniqueness of stochastic entropy solutions for stochastic balance laws with Lipschitz fluxes *

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Abstract

In this paper, we consider a stochastic balance law with a Lipschitz flux and gain the uniqueness for stochastic entropy solutions. The argument is supported by the stochastic kinetic formulation, the Itô formula and the regularization techniques. Furthermore, as an application, we derive the uniqueness of stochastic entropy solutions for stochastic porous media type equations.

Keywords: Uniqueness; Stochastic entropy solution; Stochastic kinetic formula; Itô formula

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1 Introduction

We are interested in the uniqueness of stochastic entropy solutions for the following stochastic balance law:

$$\partial_t \rho(t,x) + \mathrm{div}_x(B(\rho)) + \partial_{x_i} B_{i,j}(t,\rho) \circ \dot{W}_j(t) = A(t,x,\rho), \tag{1.1}$$

in $\Omega \times (0,\infty) \times \mathbb{R}^d$, with given initial condition:

$$\rho(t,x)|_{t=0} = \rho_0(x) \quad \text{in} \quad \Omega \times \mathbb{R}^d, \tag{1.2}$$

where $W(t) = (W_1(t), W_2(t), \ldots, W_d(t))^\top$ is an d-dimensional standard Wiener process on the classical Wiener space $(\Omega, \mathcal{F}, P, (\mathcal{F}_s)_{s\geqslant 0})$, i.e. Ω is the space of all continuous functions from $[0,\infty)$ to $\mathbb{R}^d$ with locally uniform convergence topology, $\mathcal{F}$ is the Borel σ-field, P is the Wiener measure, $(\mathcal{F}_s)_{s\geqslant 0}$ is the natural filtration generated by the coordinate process $W(t,\omega) = \omega(t)$. $\circ$ is the Stratonovich convention and

$$\left\{ \begin{array}{l} A(t,x,v) \in L^1_{loc}([0,\infty); L^1(\mathbb{R}^d_x; W^{1,1}_{loc}(\mathbb{R}_v))) + L^1_{loc}([0,\infty); L^\infty(\mathbb{R}^d_x; W^{1,1}_{loc}(\mathbb{R}_v))), \\ A(t,x,0) = 0, \quad B \in W^{1,1}_{loc}(\mathbb{R};\mathbb{R}^d), \ B_{i,j} \in L^2_{loc}([0,\infty); W^{1,2}_{loc}(\mathbb{R})), \quad 1 \leqslant i,j \leqslant d. \end{array} \right. \tag{1.3}$$

The initial function is assumed to be non-random and

$$\rho_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). \tag{1.4}$$

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Here the use of the Stratonovich differential stems from the fact that ordinary differential equations with time dependent converging Brownian motion give rise stochastic differential equations of Stratonovich's.

When $B_{i,j} = 0$ $(1 \leqslant i, j \leqslant d)$, (1.1) reduces to a deterministic partial differential equation known as the balance law

$$\partial_t \rho(t,x) + \mathrm{div}_x(B(\rho)) = A(t,x,\rho), \quad \text{in} \quad (0,\infty) \times \mathbb{R}^d. \tag{1.5}$$

The first pioneering result on the well-posedness is due to Kružkov [1]. Under the Lipschitz assumption on B and A, he obtains the existence in company with uniqueness of admissible entropy solutions for (1.6).

When $A = 0$, $B = 0$ and $(B_{i,j}(t,\rho)) = \mathrm{diag}(A_1(\rho), A_2(\rho), \ldots, A_d(\rho))$, Lions, Perthame and Souganidis [2] also study the Cauchy problem (1.1), (1.2). Under the presumption that $A \in \mathcal{C}^2$, they develop a pathwise theory for scalar conservation laws with quasilinear multiplicative rough path dependence.

Besides the stochastic quasilinear dependence, recently Feng and Nualart [3], Debussche and Vovelle [4], Chen, Ding and Karlsen [5] (also see Hofmanová [6]) put forward a theory of stochastic entropy solutions of scalar conservation laws with Itô-type, which in our setting take the form

$$\left\{ \begin{array}{l} \partial_t \rho(t,x) + \mathrm{div}_x(B(\rho)) = A(x,\rho)\dot{W}_t, \quad \text{in} \quad \Omega \times (0,\infty) \times \mathbb{R}^d, \\ \rho(t,x)|_{t=0} = \rho_0(x), \quad \text{in} \quad \Omega \times \mathbb{R}^d, \end{array} \right. \tag{1.6}$$

where $B \in \mathcal{C}^2$, A is a Lipschitz continuous function, W ia a one dimensional Wiener process. In their papers, they gain the uniqueness of stochastic entropy solutions as well as the existence.

It is remarked that all above mentioned works are concentrate their attention of stochastic entropy solutions for stochastic balance laws on $\mathcal{C}^2$-fluxes. There are relatively few papers concerned with $\mathcal{C}^1$ or Lipschitz fluxes. The aim of the present paper is raising the state of the art of the theory of uniqueness for stochastic entropy solutions of stochastic balance laws with coefficients in Lipschitz spaces to the level of the Kružkov theory for those with balance laws.

Our motivation stems from two facts. The first one comes from fluid dynamics. It is well known that the fundamental fluid dynamics models are based on Navier-Stokes equations and Euler equations. However, abundant experimental observations suggest that stochastic Navier-Stokes or Euler equations seem to be more viable models, and the stochastic balance law with the form of (1.1) can be viewed as a simple caricature of the stochastic Euler equations. In addition, stochastic balance laws like (1.1) arises as models in the theory of mean field games developed by Lasry and Lions [7-9].

The main idea is the stochastic kinetic formulation (see [2,10] for example). In fact, if $\rho \in L^\infty(\Omega; L^\infty_{loc}([0,\infty); L^\infty(\mathbb{R}^d))) \cap \mathcal{C}([0,\infty); L^1(\mathbb{R}^d \times \Omega))$ is a stochastic entropy solution of (1.1), (1.2), with B and A are smooth, then for any $v \in \mathbb{R}$, by the kinetic formulation, $u(t,x,v)$, defined by

$$u(t,x,v) = \chi_{\rho(t,x)}(v) = \left\{ \begin{array}{rl} 1, & \text{when} \ \ 0 < v < \rho, \\ -1, & \text{when} \ \ \rho < v < 0, \\ 0, & \text{otherwise}, \end{array} \right. \tag{1.7}$$

is a stochastic weak solution of the stochastic transport equation

$$\partial_t u(t,x,v) + b(v) \cdot \nabla_x u + \partial_{x_i} u \circ \dot{M}_i(t,v) + A(t,x,v)\partial_v u(t,x,v) = \partial_v m, \tag{1.8}$$

in $\Omega\times(0,\infty)\times\mathbb{R}^{d+1}$, supplied with

$$u(t,x,v)|_{t=0}=\chi_{\rho_0}(v) \ \text{ in } \ \mathbb{R}^{d+1}, \tag{1.9}$$

where m is a nonnegative measure, and

$$M_i(t,v)=\int_0^t \sigma_{i,j}(s,v)dW_j(s). \tag{1.10}$$

Thus the uniqueness of stochastic weak solutions for (1.8), (1.9) may lead to the uniqueness of stochastic entropy solutions for (1.1), (1.2), which is of particular mathematical interest and here we give a positive answer to the uniqueness of stochastic entropy solutions for (1.1), (1.2) on Lipschitz fluxes.

This paper is organized as follows. In Section 2, we give a stochastic kinetic formula for stochastic balance laws with rough coefficients. Section 3 is devoted to the uniqueness on stochastic entropy solutions for (1.1), (1.2).

As usual, the notation here is mostly standard. $C(T)$ denotes a positive constant depending only on T, whose value may change in different places. The summation convention is enforced throughout this article, wherein summation is understood with respect to repeated indices. $\mathcal{D}(\mathbb{R}^d)$ stands for the set of all smooth functions in $\mathbb{R}^d$ and supported in a compact subset in $\mathbb{R}^d$, $\mathcal{D}'(\mathbb{R}^d)$ represents its dual space. Correspondingly, $\mathcal{D}_+(\mathbb{R}^d)$ is the non-negative elements in $\mathcal{D}(\mathbb{R}^d)$. $\mathbb{N}$ denotes the set consisting of all natural numbers. a.s. is the abbreviation of "almost surely".

2 A stochastic kinetic formula

In this section, we are interested in the Cauchy problem (1.1), (1.2). For notional simplicity, we denote by $b=B'$, $\sigma_{i,j}(t,v)=\partial_v B_{i,j}(t,v)$, and we assume $\sigma_{i,j}=\sigma_{j,i}$. Moreover, we set

$$a=(a_{i,j})_{1\leqslant i,j\leqslant d}=\frac{1}{2}(\sigma_{i,j})_{1\leqslant i,j\leqslant d}(\sigma_{i,j})_{1\leqslant i,j\leqslant d}. \tag{2.1}$$

Initially, we need give some notions.

Definition 2.1 *$\rho\in L^\infty(\Omega;L^\infty_{loc}([0,\infty);L^\infty(\mathbb{R}^d)))\cap\mathcal{C}([0,\infty);L^1(\mathbb{R}^d\times\Omega))$ is a stochastic weak solution of (1.1) and (1.2), if for any $\varphi\in\mathcal{D}(\mathbb{R}^d)$, $\int_{\mathbb{R}^d}\rho(t,x)\varphi(x)dx$ is an $\mathcal{F}_t$ semi-martingale and with probability one, the below identity*

$$\begin{aligned}&\int_{\mathbb{R}^d}\varphi(x)\rho(t,x)dx-\int_0^t\int_{\mathbb{R}^d}B(\rho)\cdot\nabla_x\varphi(x)dxds-\int_0^t\circ dW_j(s)\int_{\mathbb{R}^d}\partial_{x_i}\varphi(x)B_{i,j}(s,\rho)dx\\ =\;&\int_{\mathbb{R}^d}\varphi(x)\rho_0(x)dx+\int_0^t\int_{\mathbb{R}^d}A(s,x,\rho)\varphi(x)dxds,\end{aligned} \tag{2.2}$$

holds true, for all $t\in[0,\infty)$.

Remark 2.1 *Obviously, (2.2) admits an equivalent representation:*

$$\int_0^\infty\int_{\mathbb{R}^d}\partial_t\psi(t,x)\rho(t,x)dxdt+\int_{\mathbb{R}^d}\rho_0(x)\psi(0,x)dx+\int_0^\infty\int_{\mathbb{R}^d}B(\rho)\cdot\nabla\psi(t,x)dxdt$$

$$= \quad -\int_0^\infty \circ dW_j(t)\int_{\mathbb{R}^d} B_{i,j}(t,\rho)\partial_{x_i}\psi(t,x)dx - \int_0^\infty\int_{\mathbb{R}^d} A(t,x,\rho)\psi(t,x)dxdt,$$

for any $\psi(t,x)\in\mathcal{D}([0,\infty)\times\mathbb{R}^d)$*, and for almost all* $\omega\in\Omega$*. Especially,*

$$\int_0^T\int_{\mathbb{R}^d}\partial_t\tilde{\psi}(t,x)\rho(t,x)dxdt + \int_{\mathbb{R}^d}\rho_0(x)\tilde{\psi}(0,x)dx + \int_0^T\int_{\mathbb{R}^d} B(\rho)\cdot\nabla\tilde{\psi}(t,x)dxdt$$
$$= \quad -\int_0^T \circ dW_j(t)\int_{\mathbb{R}^d} B_{i,j}(t,\rho)\partial_{x_i}\tilde{\psi}(t,x)dx - \int_0^T\int_{\mathbb{R}^d} A(t,x,\rho)\tilde{\psi}(t,x)dxdt,$$

for any $\tilde{\psi}(t,x)\in\mathcal{D}([0,T)\times\mathbb{R}^d)$.

Definition 2.2 *A stochastic weak solution of (1.1), (1.2) is a stochastic entropy solution, if for any* $\eta\in\Xi$,

$$\partial_t\eta(\rho) + div(Q(\rho)) + \partial_{x_i}Q_{i,j}(t,\rho)\circ\dot{W}_j(t) \leqslant h(t,x,\rho),\ \ P-a.s.\ \omega\in\Omega, \tag{2.3}$$

in the sense of distributions, i.e., for any $\psi(t,x)\in\mathcal{D}_+([0,\infty)\times\mathbb{R}^d)$ *and for almost all* $\omega\in\Omega$

$$\int_0^\infty dt\int_{\mathbb{R}^d}\partial_t\psi\eta(\rho)dx + \int_0^\infty\int_{\mathbb{R}^d} Q(\rho)\cdot\nabla_x\psi dxdt + \int_0^\infty \circ dW_j(t)\int_{\mathbb{R}^d}\partial_{x_i}\psi Q_{i,j}(t,\rho)dx$$
$$\geqslant \quad -\int_{\mathbb{R}^d}\psi(0,x)\eta(\rho_0)dx - \int_0^\infty\int_{\mathbb{R}^d} h(t,x,\rho)\psi(t,x)dxdt,$$

where

$$Q(\rho) = \int^\rho \eta'(v)b(v)dv,\ \ Q_{i,j}(t,\rho) = \int^\rho \eta'(v)\sigma_{i,j}(t,v)dv,\ \ h(t,x,\rho) = A(t,x,\rho)\eta'(\rho), \tag{2.4}$$

and

$$\Xi = \{c_0\rho + \sum_{k=1}^n c_k|\rho-\rho_k|,\quad c_0,\rho_k,c_k\in\mathbb{R}\ \ \text{are constants}\}.$$

Remark 2.2 *(i)* $\eta\in\Xi$*, so* η *is convex and thus* $\eta'(v)$ *is legitimate for almost all* $v\in\mathbb{R}$*. But to make* $A(t,x,\rho)\eta'(\rho)$ *well-defined, we need to choose a particular modification for* $\eta'(\rho)$*. Here in our mind, we take* $\partial_\rho|\rho-\rho_k| = sign(\rho-\rho_k)$.

(ii) If ρ, B, A *and* $B_{i,j}$ *are smooth, for any convex function* η*, then*

$$\partial_t\eta(\rho) + div_x(Q(\rho)) + \partial_{x_i}Q_{i,j}(t,\rho)\circ\dot{W}_j(t) = h(t,x,\rho),$$

with Q*,* $Q_{i,j}$ *and* h *given by (2.4). In general, (1.1) should serve as the* $\varepsilon\to 0$ *limit of the equation*

$$\partial_t\rho_\varepsilon(t,x) + div_x(B(\rho_\varepsilon)) + \partial_{x_i}B_{i,j}(t,\rho)\circ\dot{W}_j(t) - \varepsilon\Delta\rho_\varepsilon(t,x) = A(t,x,\rho_\varepsilon).$$

Observe that now

$$\partial_t\eta(\rho_\varepsilon) + div_x(Q(\rho_\varepsilon)) + \partial_{x_i}Q_{i,j}(t,\rho)\circ\dot{W}_j(t) - \varepsilon\Delta\eta(\rho_\varepsilon) \leqslant h(t,x,\rho_\varepsilon),$$

so the vanishing viscosity limit should make above inequality to conserve the same sigma. Thus Definition 2.2 is reasonable.

(iii) When $\eta(\rho,\bar{\rho}) = |\rho-\bar{\rho}|$*, then*

$$Q(\rho,\bar{\rho}) = sign(\rho-\bar{\rho})[B(\rho)-B(\bar{\rho})],\quad Q_{i,j}(t,\rho,\bar{\rho}) = sign(\rho-\bar{\rho})[B_{i,j}(t,\rho)-B_{i,j}(t,\bar{\rho})],$$

where $\bar{\rho}$ *is a parameter taking values in* $\mathbb{R}$.

Before founding the uniqueness of stochastic entropy solutions, we need establish the equivalence between stochastic entropy solutions of (1.1), (1.2) and stochastic weak solutions for (1.8), (1.9).

Theorem 2.1 ***(Stochastic kinetic formula)*** *Assume that A B and $B_{i,j}$ fulfill the assumption (1.3).*

(i) Let ρ be a stochastic entropy solution of (1.1), (1.2) and set $u(t,x,v)=\chi_{\rho(t,x)}(v)$. Then

$$u\in L^{\infty}(\Omega;L^{\infty}_{loc}([0,\infty);L^{\infty}(\mathbb{R}^d_x;L^1(\mathbb{R}_v))))\cap\mathcal{C}([0,\infty);L^1(\mathbb{R}^d_x\times\mathbb{R}_v\times\Omega)), \tag{2.5}$$

and it is a stochastic weak solution of the linear stochastic transport problem (1.8), (1.9) with $M_i(t,v)$ given by (1.10), and $0\leqslant m\in L^1(\Omega;\mathcal{D}'([0,\infty)\times\mathbb{R}^d_x\times\mathbb{R}_v))$, satisfying, for any $T>0$, and for almost all $\omega\in\Omega$, m is bounded on $[0,T]\times\mathbb{R}^d\times\mathbb{R}$, supported in $[0,T]\times\mathbb{R}^d\times[-K,K]$ ($K=\|\rho\|_{L^{\infty}((0,T)\times\mathbb{R}^d\times\Omega)}$), continuous in t, here the continuous is interpreted

$$m([0,s]\times\mathbb{R}^{d+1})\rightarrow m([0,t]\times\mathbb{R}^{d+1}),\quad as\quad s\rightarrow t. \tag{2.6}$$

Here u is called a stochastic weak solution of (1.8), (1.9), if for any $\phi\in\mathcal{D}(\mathbb{R}^{d+1})$, with probability one,

$$\begin{aligned}&\int_{\mathbb{R}^{d+1}}\phi(x,v)u(t,x,v)dxdv-\int_0^t\int_{\mathbb{R}^{d+1}}b(v)\cdot\nabla_x\phi(x,v)dxdvds\\=&\int_{\mathbb{R}^{d+1}}\phi(x,v)u_0(x,v)dxdv+\int_0^t\int_{\mathbb{R}}M_i(\circ ds,dv)\int_{\mathbb{R}^d}\partial_{x_i}\phi(x,v)u(s,x,v)dx\\&+\int_0^t\int_{\mathbb{R}^{d+1}}\partial_v[A(s,x,v)\phi(x,v)]u(s,x,v)dxdvds-\int_0^t\int_{\mathbb{R}^{d+1}}\partial_v\phi(x,v)m(dx,dv,ds),\end{aligned}$$

is legitimate, for all $t\in[0,\infty)$.

(ii) Let $u\in L^{\infty}(\Omega;L^{\infty}_{loc}([0,\infty);L^{\infty}(\mathbb{R}^d_x;L^1(\mathbb{R}_v))))\cap\mathcal{C}([0,\infty);L^1(\mathbb{R}^d_x\times\mathbb{R}_v\times\Omega))$ be a stochastic weak solution of (1.8), (1.9), with m meeting above properties. Then

$$\rho(t,x)=\int_{\mathbb{R}}u(t,x,v)dv\in L^{\infty}(\Omega;L^{\infty}_{loc}([0,\infty);L^{\infty}(\mathbb{R}^d)))\cap\mathcal{C}([0,\infty);L^1(\mathbb{R}^d\times\Omega)), \tag{2.7}$$

and it is a stochastic entropy solution of (1.1), (1.2).

Before proving above kinetic formula, we give two lemmas which will serve us well later.

Lemma 2.1 *(1.8) has the following equivalent representation:*

$$\partial_t u(t,x,v)+b(v)\cdot\nabla_x u+\partial_{x_i}u\dot{M}_i(t,v)-a_{i,j}(t,v)\partial^2_{x_i,x_j}u+A(t,x,v)\partial_v u=\partial_v m, \tag{2.8}$$

Proof. Clearly, it suffices to show: for any $\phi\in\mathcal{D}(\mathbb{R}^{d+1})$, and for all $t\in[0,\infty)$,

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^{d+1}}\partial_{x_i}\phi(x,v)u(s,x,v)dxM_i(\circ ds,dv)\\=&\int_0^t\int_{\mathbb{R}^{d+1}}\partial_{x_i}\phi(x,v)u(s,x,v)dxM_i(ds,dv)+\int_0^t ds\int_{\mathbb{R}^{d+1}}a_{i,j}(s,v)\partial^2_{x_i,x_j}\phi udxdv.\end{aligned}$$

But on the other hand,

$$\int_0^t\int_{\mathbb{R}^{d+1}}\partial_{x_i}\phi(x,v)u(s,x,v)dxM_i(\circ ds,dv)$$

$$= \int_0^t \int_{\mathbb{R}^{d+1}} \partial_{x_i}\phi(x,v)u(s,x,v)dxM_i(ds,dv) + \frac{1}{2}\int_{\mathbb{R}}[\int_{\mathbb{R}^d} \partial_{x_i}\varphi(x,v)u(\cdot,x,v)dx, M_i(\cdot,v)]_t dv,$$

where $[\cdot,\cdot]_t$ denotes the joint quadratic variation, thus it is sufficient to demonstrate

$$\int_{\mathbb{R}}[\int_{\mathbb{R}^d} \partial_{x_i}\phi(x,v)u(\cdot,x,v)dx, M_i(\cdot,v)]_t dv = 2\int_0^t ds \int_{\mathbb{R}^{d+1}} a_{i,j}(s,v)\partial^2_{x_i,x_j}\phi u dxdv.$$

Note that whichsoever (1.8) or (2.8) holds, then for any $\phi \in \mathcal{D}(\mathbb{R}^{d+1})$, and for all $t \in [0,\infty)$, the martingale part of $\int_{\mathbb{R}^d} \partial_{x_i}\varphi(x,v)u(t,x,v)dx$ is given by

$$\int_0^t \int_{\mathbb{R}^d} \partial^2_{x_i,x_j}\phi(x,v)u(s,x,v)dxM_j(ds,v).$$

Therefore

$$\begin{aligned}
& \int_{\mathbb{R}}[\int_{\mathbb{R}^d} \partial_{x_i}\phi(x,v)u(\cdot,x,v)dx, M_i(\cdot,v)]_t dv \\
= & \int_{\mathbb{R}}[\int_0^{\cdot} M_j(ds,v)\int_{\mathbb{R}^d} \partial^2_{x_i,x_j}\phi(x,v)u(s,x,v)dx, M_i(\cdot,v)]_t dv \\
= & \int_{\mathbb{R}} dv \int_0^t ds \int_{\mathbb{R}^d} \partial^2_{x_i,x_j}\phi(x,v)u(s,x,v)\sigma_{i,k}(s,v)\sigma_{j,k}(s,v)dx \\
= & \ 2\int_0^t ds \int_{\mathbb{R}^{d+1}} a_{i,j}(s,v)\partial^2_{x_i,x_j}\phi(x,v)u(s,x,v)dxdv.
\end{aligned}$$

Lemma 2.2 *For any $p \in [1,\infty]$, we have the following embedding:*

$$L^p(\mathbb{R}^d; W^{1,1}_{loc}(\mathbb{R})) \hookrightarrow L^p(\mathbb{R}^d; \mathcal{C}(\mathbb{R})).$$

Proof. Clearly, $W^{1,1}_{loc}(\mathbb{R}) \hookrightarrow \mathcal{C}(\mathbb{R})$, so for any $f \in L^p(\mathbb{R}^d; W^{1,1}_{loc}(\mathbb{R}))$, $f(x,\cdot) \in \mathcal{C}(\mathbb{R})$ for a.s. $x \in \mathbb{R}^d$. Let $-\infty < a < b < \infty$ be two real numbers, then
when $p < \infty$,

$$\begin{aligned}
\|f\|^p_{L^p(\mathbb{R}^d;L^\infty(a,b))} &= \int_{\mathbb{R}^d} \|f(x,\cdot)\|^p_{L^\infty(a,b)}dx \\
&= \int_{\mathbb{R}^d} \|\int_a^{\cdot} \partial_v f(x,v)dv + f(x,a)\|^p_{L^\infty(a,b)}dx \\
&\leqslant 2^{p-1}\Big[\int_{\mathbb{R}^d}[\int_a^b |\partial_v f(x,v)|dv]^p dx + \int_{\mathbb{R}^d} |f(x,a)|^p dx\Big] \\
&< \infty,
\end{aligned}$$

when $p = \infty$,

$$\begin{aligned}
|f(x,v)| &= |\int_a^v \partial_y f(x,y)dy + f(x,a)| \\
&\leqslant \int_a^v |\partial_y f(x,y)|dy + |f(x,a)|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_a^b |\partial_y f(x,y)|dy + |f(x,a)| \\
&< \infty
\end{aligned}
$$

for almost all $x \in \mathbb{R}^d$, and all $v \in [a,b]$, which hints

$$L^p(\mathbb{R}^d; W^{1,1}_{loc}(\mathbb{R})) \hookrightarrow L^p(\mathbb{R}^d; L^\infty_{loc}(\mathbb{R})).$$

Thus the desired result follows.

Proof of Theorem 2.1. For any $\alpha_1, \alpha_2 \in \mathbb{R}$, notice that

$$\int_{\mathbb{R}} |\chi_{\alpha_1}(v) - \chi_{\alpha_2}(v)|dv = |\alpha_1 - \alpha_2|,$$

so (2.7) implies (2.5), and the vice versa is clear, we need to survey the rest of (i) and (ii).

(i) Suppose that ρ is a stochastic entropy solution of (1.1), (1.2) fulfilling the statement in (i), for any $v \in \mathbb{R}$, it renders that

$$\partial_t \eta(\rho,v) + \mathrm{div}_x Q(\rho,v) + \partial_{x_i} Q_{i,j}(t,\rho,v) \circ \dot{W}_j(t) = \mathrm{sign}(\rho - v)A(t,x,\rho) - 2m, \tag{2.9}$$

where

$$\left\{\begin{array}{l} \eta(\rho,v) = |\rho - v| - |v|, \\ Q(\rho,v) = \mathrm{sign}(\rho - v)[B(\rho) - B(v)] - \mathrm{sign} v B(v), \\ Q_{i,j}(t,\rho,v) = \mathrm{sign}(\rho - v)[B_{i,j}(t,\rho) - B_{i,j}(t,v)] - \mathrm{sign} v B_{i,j}(t,v), \\ m \text{ is a nonnegative measure on } [0,\infty) \times \mathbb{R}^d \times \mathbb{R}, \end{array}\right. \tag{2.10}$$

for almost all $\omega \in \Omega$.

On account of (1.3), if one differentiates (2.9) in v in distributional sense, then

$$\left\{\begin{array}{l} \partial_v \eta(\rho,v) = -2u(t,x,v), \\ \partial_v Q(\rho,v) = -2b(v)u(t,x,v), \\ \partial_v Q_{i,j}(t,\rho,v) = -2\sigma_{i,j}(t,v)u(t,x,v), \\ \partial_v \mathrm{sign}(\rho - v)A(t,x,\rho) = 2\partial_v u(t,x,v)A(t,x,v). \end{array}\right. \tag{2.11}$$

Thus one derives the identity (1.8) in the sense of distributions for almost all $\omega \in \Omega$.

In fact, if one takes the last identity in (2.11) for an example, then for any $\phi_1 \in \mathcal{D}(\mathbb{R})$,

$$\langle \partial_v \mathrm{sign}(\rho - v)A(t,x,\rho),\ \phi_1 \rangle = -2\phi_1(\rho)A(t,x,\rho).$$

On the other hand

$$\langle \partial_v u(t,x,v)A(t,x,v),\ \phi_1 \rangle = -\langle u(t,x,v),\ \partial_v(\phi_1(v)A(t,x,v)) \rangle.$$

Note that

$$\int_{\mathbb{R}} g'(v)u(t,x,v)dv = g(\rho(t,x)) - g(0), \text{ for any } g \in W^{1,1}_{loc}(\mathbb{R}), \tag{2.12}$$

and $A(t,x,0) = 0$, it follows that

$$\langle \partial_v u(t,x,v)A(t,x,v),\ \phi_1 \rangle = -\phi_1(\rho)A(t,x,\rho),$$

thus $\partial_v \mathrm{sign}(\rho - v)A(t,x,\rho) = 2\partial_v u(t,x,v)A(t,x,v)$.

In view of that ρ is bounded local-in-time, from (2.9) and (2.10), for any fixed $T>0$, and almost all $\omega \in \Omega$, m is supported in $[0,T]\times \mathbb{R}^d \times [-K,K]$, with $K = \|\rho\|_{L^\infty((0,T)\times\mathbb{R}^d\times\Omega)}$. Accordingly, it remains to examine that m is bounded and continuous in t.

Since $m \geqslant 0$ and it is supported in a compact subset in v, we obtain

$$\begin{aligned} 0 &\leqslant \langle m,\ \psi\otimes 1\rangle_{t,x,v} \\ &= -\langle \partial_t u + b(v)\cdot\nabla_x u + A\partial_v u + \partial_{x_i} u \circ \dot{M}_i(t,v),\ \psi\otimes v\rangle_{t,x,v}, \end{aligned}$$

for any $\psi \in \mathcal{D}_+([0,\infty)\times\mathbb{R}^d)$, and for almost all $\omega\in\Omega$.

With the aid of Lemma 2.1, then

$$\begin{aligned} 0 \leqslant \langle m,\ \psi\otimes 1\rangle_{t,x,v} &= -\langle \partial_t u + b(v)\cdot\nabla_x u + A(t,x,v)\partial_v u(t,x,v),\ \psi\otimes v\rangle_{t,x,v} \\ &\quad -\langle \sigma_{i,j}(t,v)\partial_{x_i} u \dot{W}_j(t) - a_{i,j}(t,v)\partial^2_{x_i,x_j} u,\ \psi\otimes v\rangle_{t,x,v}. \end{aligned} \tag{2.13}$$

Thanks to (2.12), one computes from (2.13) that

$$\begin{aligned} &-\langle \partial_t u + b(v)\cdot\nabla_x u + A\partial_v u + \sigma_{i,j}(t,v)\partial_{x_i} u\dot{W}_j(t) - a_{i,j}(t,v)\partial^2_{x_i,x_j} u,\ \psi\otimes v\rangle_{t,x,v} \\ = &\ \frac{1}{2}\int_0^T\int_{\mathbb{R}^d} \partial_t\psi(t,x)\rho^2 dxdt + \frac{1}{2}\int_{\mathbb{R}^d}\psi(0,x)\rho_0^2(x)dx + \int_0^T\int_{\mathbb{R}^d}\rho A(t,x,\rho)\psi(t,x)dxdt \\ &+\int_0^T\int_{\mathbb{R}^d}\Big[\rho(t,x)B(\rho(t,x)) - \int_0^{\rho(t,x)} B(v)dv\Big]\cdot\nabla_x\psi(t,x)dxdt \\ &+\int_0^T\int_{\mathbb{R}^d}\Big[A_{i,j}(t,\rho(t,x))\rho(t,x) - \int_0^{\rho(t,x)} A_{i,j}(t,v)dv\Big]\partial^2_{x_i,x_j}\psi(t,x)dxdt \\ &+\int_0^T\int_{\mathbb{R}^d}\Big[B_{i,j}(t,\rho)\rho(t,x) - \int_0^{\rho(t,x)} B_{i,j}(t,v)dv\Big]\partial_{x_i}\psi(t,x)dxdW_j(t),\ \ P-a.s.\ \omega\in\Omega, \end{aligned}$$

for any $T>0$ and $\psi\in\mathcal{D}_+([0,T)\times\mathbb{R}^d)$, where $\partial_v A_{i,j}(t,v) = a_{i,j}(t,v)$.

On account of hypotheses (1.3), in view of Lemma 2.2, this leads to

$$\begin{aligned} &-\langle \partial_t u + b(v)\cdot\nabla_x u + A\partial_v u + \sigma_{i,j}(t,v)\partial_{x_i} u\dot{W}_j(t) - a_{i,j}(t,v)\partial^2_{x_i,x_j} u,\ \psi\otimes v\rangle_{t,x,v} \\ \leqslant &\ \frac{1}{2}\int_0^T\int_{\mathbb{R}^d}\partial_t\psi\rho^2 dxdt + \frac{1}{2}\int_{\mathbb{R}^d}\psi(0,x)\rho^2 dx + C(T)\int_0^T\int_{\mathbb{R}^d}\tilde{a}(t,x)\rho^2\psi(t,x)dxdt \\ &+ C(T)\Big[\int_0^T\int_{\mathbb{R}^d}|\rho(t,x)||\nabla_x\psi(t,x)|dxdt + \int_0^T\int_{\mathbb{R}^d}\tilde{a}_{i,j}(t)|\rho(t,x)||\partial^2_{x_i,x_j}\psi(t,x)|dxdt\Big] \\ &+\int_0^T\int_{\mathbb{R}^d}\Big[B_{i,j}(t,\rho)\rho - \int_0^{\rho(t,x)} B_{i,j}(t,v)dv\Big]\partial_{x_i}\psi(t,x)dxdW_j(t),\ \ P-a.s.\ \omega\in\Omega, \end{aligned} \tag{2.14}$$

where

$$\tilde{a}(t,x) = \sup_{v\in[-K,K]}|A(t,x,v)| \in L^1_{loc}([0,\infty);L^1(\mathbb{R}^d)) + L^1_{loc}([0,\infty);L^\infty(\mathbb{R}^d)),$$

$$\tilde{a}_{i,j}(t) \;=\; \sup_{v\in[-K,K]} |A_{i,j}(t,v)| \in L^1_{loc}([0,\infty)).$$

Obviously, (2.14) holds ad hoc for $\psi(t,x)=\psi_1(t)\theta_n(x)$, where $\psi_1\in\mathcal{D}_+([0,T)),\ \theta\in\mathcal{D}_+(\mathbb{R}^d)$,

$$\theta_n(x)=\theta(\frac{x}{n}),\ \theta(x)=\begin{cases}1, & \text{when } |x|\leqslant 1,\\ 0, & \text{when } |x|>2,\end{cases} \tag{2.15}$$

and for this fixed n, by an approximation demonstration, one can fetch

$$\psi_1(t)=\begin{cases}1, & t\in[0,T-\frac{1}{n}],\\ -n(t-T), & t\in(T-\frac{1}{n},T],\\ 0, & t\in(T,\infty).\end{cases}$$

Observing that the Itô isometry,

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^d}[B_{i,j}(t,\rho)\rho-\int_0^{\rho(t,x)}B_{i,j}(t,v)dv]\partial_{x_i}\psi(t,x)dxdW_j(t)\Big]^2\\
=\;&\int_0^T\mathbb{E}\Big[\int_{\mathbb{R}^d}[B_{i,j}(t,\rho)\rho-\int_0^{\rho(t,x)}B_{i,j}(t,v)dv]\partial_{x_i}\psi(t,x)dx\Big]^2dt\\
\leqslant\;& C(T)\mathbb{E}\int_0^T\tilde{b}^2_{i,j}(t)\Big[\int_{\mathbb{R}^d}|\rho(t,x)||\partial_{x_i}\psi(t,x)|dx\Big]^2dt,
\end{aligned}$$

where

$$\tilde{b}_{i,j}(t) \;=\; \sup_{v\in[-K,K]} |B_{i,j}(t,v)| \in L^2_{loc}([0,\infty)).$$

Thus we gain from (2.13) and (2.14) by letting $n\to\infty$, that

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d_x}\int_{\mathbb{R}_v}m(dt,dx,dv)\\
\leqslant\;&\frac{1}{2}\Big[\int_{\mathbb{R}^d}\rho_0^2dx-\int_{\mathbb{R}^d}\rho^2(T,x)dx\Big]+C(T)\int_0^T\int_{\mathbb{R}^d}\tilde{a}(t,x)\rho^2(t,x)dxdt,
\end{aligned}\tag{2.16}$$

for $P-a.s.\ \omega\in\Omega$, which suggests that m is bounded on $[0,T]\times\mathbb{R}^d\times\mathbb{R}$, for any given $T>0$, and $m\in L^1(\Omega;\mathcal{D}'([0,\infty)\times\mathbb{R}^{d+1}))$.

Specially, when $T\to 0$, we obtain

$$\lim_{T\to 0}\int_0^T\int_{\mathbb{R}^d_x}\int_{\mathbb{R}_v}m(dt,dx,dv)=0,\quad P-a.s.\ \omega\in\Omega.$$

The arguments employed above for 0 and T adapted to any $0\leqslant s,t<\infty$ now, yields that

$$\lim_{t\to s}\int_s^t\int_{\mathbb{R}^d_x}\int_{\mathbb{R}_v}m(dr,dx,dv)=0,$$

which hints m is continuous in t. By Remark 2.1, u is a stochastic weak solution of (1.8), (1.9).

(ii) Let us show the reverse fact. Given $\epsilon>0$ and $\bar{\rho}\in\mathbb{R}$, set

$$\eta_\epsilon(t,\bar{\rho})=(\sqrt{(t-\bar{\rho})^2+\epsilon^2}-\epsilon)-|\bar{\rho}|\in\mathcal{C}^2(\mathbb{R}),$$

then η_ϵ is convex, $\eta'_\epsilon(t,\bar{\rho}) \in \mathcal{C}_b(\mathbb{R})$, and

$$\eta_\epsilon(t,\bar{\rho}) \longrightarrow |t-\bar{\rho}| - |\bar{\rho}| \ \text{ as } \ \epsilon \longrightarrow 0.$$

In a consequence of $u(t,x,v)$ solving (1.8) and (1.9), by Remark 2.1, it follows that

$$\begin{aligned}\langle \partial_v m,\ \psi\eta'_\epsilon(v,\bar{\rho})\xi_k(v)\rangle_{t,x,v} &= \langle \partial_t u + \partial_{x_i} u \circ \dot{M}_i(t,v),\ \psi\eta'_\epsilon(v,\bar{\rho})\xi_k(v)\rangle_{t,x,v} \\ &\quad + \langle b(v)\cdot\nabla_x u + A\partial_v u,\ \psi\eta'_\epsilon(v,\bar{\rho})\xi_k(v)\rangle_{t,x,v} \\ &= \langle \partial_t u + \sigma_{i,j}(t,v)\partial_{x_i} u \circ \dot{W}_j(t),\ \psi\eta'_\epsilon(v,\bar{\rho})\xi_k(v)\rangle_{t,x,v} \\ &\quad + \langle b(v)\cdot\nabla_x u + A\partial_v u,\ \psi\eta'_\epsilon(v,\bar{\rho})\xi_k(v)\rangle_{t,x,v},\end{aligned} \tag{2.17}$$

for any $\psi \in \mathcal{D}_+([0,\infty)\times\mathbb{R}^d)$, $\xi \in \mathcal{D}_+(\mathbb{R})$, where

$$\xi_k(v) = \xi(\frac{v}{k}),\ 0 \leqslant \xi \leqslant 1,\ \xi(v) = \begin{cases} 1, & \text{when } |v| \leqslant 1, \\ 0, & \text{when } |v| \geqslant 2. \end{cases} \tag{2.18}$$

Applying the partial integration, one deduces

$$\begin{aligned}&\lim_{k\to\infty} \langle \partial_v m,\ \psi\eta'_\epsilon(v,\bar{\rho})\xi_k\rangle_{t,x,v} \\ = \ & -\lim_{k\to\infty} \langle m,\ \psi[\eta''_\epsilon(v,\bar{\rho})\xi_k + \eta'_\epsilon(v,\bar{\rho})\xi'_k]\rangle_{t,x,v} \\ \leqslant \ & 0,\ P-a.s.\ \omega\in\Omega,\end{aligned} \tag{2.19}$$

for m yields the properties stated in Theorem 2.1 (i).

Upon using (2.12) and (2.19), from (2.17), we derive

$$\begin{aligned}&\int_0^\infty dt \int_{\mathbb{R}^d} \partial_t\psi(t,x)[\eta_\epsilon(\rho,\bar{\rho}) - \eta_\epsilon(0,\bar{\rho})]dx + \int_0^\infty dt \int_{\mathbb{R}^d} Q_\epsilon(\rho,\bar{\rho})\cdot\nabla_x\psi dx \\ \geqslant \ & -\int_{\mathbb{R}^d} \psi(0,x)[\eta_\epsilon(\rho_0,\bar{\rho}) - \eta_\epsilon(0,\bar{\rho})]dx - \int_0^\infty dt \int_{\mathbb{R}^d} \eta'_\epsilon(\rho,\bar{\rho})A(t,x,\rho)\psi(t,x)dx \\ & -\int_0^\infty \circ dW_j \int_{\mathbb{R}^d} \partial_{x_i}\psi Q^\epsilon_{i,j}(t,\rho,\bar{\rho})dx,\end{aligned} \tag{2.20}$$

by taking k to infinity, here

$$Q_\epsilon(\rho,\bar{\rho}) = \int_{\mathbb{R}} b(v)\eta'_\epsilon(v,\bar{\rho})u(t,x,v)dv, \quad Q^\epsilon_{i,j}(t,\rho,\bar{\rho}) = \int_{\mathbb{R}} \sigma_{i,j}(t,v)\eta'_\epsilon(v,\bar{\rho})u(t,x,v)dv.$$

On the other hand

$$\lim_{\epsilon\to 0} \eta'_\epsilon(v,\bar{\rho}) = \mathrm{sign}(v-\bar{\rho})$$

and

$$\lim_{\epsilon\to 0} Q_\epsilon(\rho,\bar{\rho}) = \mathrm{sign}(\rho-\bar{\rho})[B(\rho)-B(\bar{\rho})] - \mathrm{sign}\bar{\rho}[B(\bar{\rho})-B(0)],$$

$$\lim_{\epsilon\to 0} Q^\epsilon_{i,j}(t,\rho,\bar{\rho}) = \mathrm{sign}(\rho-\bar{\rho})[B_{i,j}(t,\rho)-B_{i,j}(t,\bar{\rho})] - \mathrm{sign}\bar{\rho}[B_{i,j}(t,\bar{\rho})-B_{i,j}(t,0)],$$

for a.s. $(\omega,t,x)\in\Omega\times[0,\infty)\times\mathbb{R}^d$.

If one lets ϵ approach to zero in (2.20), we attain the inequality (2.3), thus ρ is a stochastic entropy solution.

Remark 2.3 *(i) Our proof for Theorem 2.1 is inspired by Theorem 1 in [10], but the demonstration here appears to be finer, and for more details, one can see [10] and the references cited there.*

(ii) Until now we are not clear enough how to extend the present result to more general vector-valued function ρ, for the proof here relying upon very much 'scalar features'. But for results on the deterministic 2×2 hyperbolic system of isentropic gas dynamics in both Eulerian or Lagrangian variables, one can see [11].

We are now in a position to give our main result on uniqueness for stochastic entropy solutions to (1.1), (1.2).

3 Uniqueness of stochastic entropy solutions

Before founding the uniqueness, we need two lemmas below, the first one follows from DiPerna and Lions [12], and the proof is analogue, we only give the details for the second one.

Lemma 3.1 *Let $E \in L^{p_1}(\Omega; L^{p_2}(0,T; W^{1,\alpha}_{loc}(\mathbb{R}^n;\mathbb{R}^n)))$, $F \in L^{q_1}(\Omega; L^{q_2}(0,T; L^{\beta}_{loc}(\mathbb{R}^n)))$, with $1 \leqslant p_1, p_2, q_1, q_2, \alpha, \beta \leqslant \infty$, $T \in (0,\infty)$. Then*

$$(E\cdot\nabla F)*\tilde{\varrho}_{\varepsilon_1} - E\cdot\nabla(F*\tilde{\varrho}_{\varepsilon_1}) \longrightarrow 0 \quad \text{in} \quad L^{r_1}(\Omega; L^{r_2}(0,T; L^{\gamma}_{loc}(\mathbb{R}^n))) \ \text{as} \ \varepsilon_1 \to 0,$$

where $1 \leqslant \gamma, r_1, r_2 < \infty$, satisfying

$$\frac{1}{\alpha}+\frac{1}{\beta}\leqslant\frac{1}{\gamma}, \quad \frac{1}{p_1}+\frac{1}{q_1}\leqslant\frac{1}{r_1}, \quad \frac{1}{p_2}+\frac{1}{q_2}\leqslant\frac{1}{r_2},$$

$n \in \mathbb{N}$, and

$$\tilde{\varrho}_{\varepsilon_1} = \frac{1}{{\varepsilon_1}^n}\tilde{\varrho}(\frac{\cdot}{\varepsilon_1}) \ \text{ with } \tilde{\varrho}\in\mathcal{D}_+(\mathbb{R}^n), \quad \int_{\mathbb{R}^n}\tilde{\varrho}(y)dy = 1, \quad \varepsilon_1 > 0.$$

And when $n = d$, we set $\tilde{\varrho}$ by ϱ_1.

Lemma 3.2 *Let $f \in L^2(\Omega; L^2_{loc}([0,\infty)))$, then*

$$[\int_0^{\cdot} f(s)dW_s * \varrho_{2,\varepsilon_2}](t) \longrightarrow \int_0^t f(s)dW_s, \quad \text{in} \ \ L^2(\Omega; L^2_{loc}([0,\infty))) \ \ \text{as} \ \ \varepsilon_2 \to 0,$$

where W_t is a one dimensional standard Wiener process, and

$$\varrho_{2,\varepsilon_2} = \frac{1}{\varepsilon_2}\varrho_2(\frac{\cdot}{\varepsilon_2}) \quad \varrho_2 \in \mathcal{D}_+(\mathbb{R}), \ \int_{\mathbb{R}}\varrho_2(t)dt = 1, \ supp\varrho_2 \subset (-1,0).$$

Proof. In fact, for any $T \in (0,\infty)$, then

$$\begin{aligned} & \mathbb{E}\int_0^T \Big|[\int_0^{\cdot} f(s)dW_s * \varrho_{2,\varepsilon_2}](t) - \int_0^t f(s)dW_s\Big|^2 dt \\ = \ & \mathbb{E}\int_0^T \Big|\int_{\mathbb{R}}\varrho_{2,\varepsilon_2}(s)ds\int_0^{t-s} f(r)dW_r - \int_0^t f(r)dW_r\Big|^2 dt \\ = \ & \mathbb{E}\int_0^T \Big|\int_{-1}^0\varrho_2(s)ds\int_0^{t-\varepsilon_2 s} f(r)dW_r - \int_0^t f(r)dW_r\Big|^2 dt \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\int_0^T \Big| \int_{-1}^0 \varrho_2(s)ds \int_t^{t-\varepsilon_2 s} f(r)dW_r \Big|^2 dt \\
&\leqslant \int_0^T \mathbb{E} \sup_{s\in[0,1]} \Big| \int_t^{t+\varepsilon_2 s} f(r)dW_r \Big|^2 dt. \qquad (3.1)
\end{aligned}
$$

For $f \in L^2(\Omega; L^2_{loc}([0,\infty))$, thus the stochastic process $\{\int_0^t f(r)dW_r,\ t \geqslant 0\}$ is a martingale. With the help of Doob's inequality and the Itô isometry, from (3.1), one obtains

$$
\begin{aligned}
& \mathbb{E}\int_0^T \Big| [\int_0^{\cdot} f(s)dW_s * \varrho_{2,\varepsilon_2}](t) - \int_0^t f(s)dW_s \Big|^2 dt \\
\leqslant\ & 4\int_0^T \sup_{0\leqslant s\leqslant 1} \mathbb{E}\Big| \int_t^{t+\varepsilon_2 s} f(r)dW_r \Big|^2 dt \\
=\ & 4\int_0^T \int_t^{t+\varepsilon_2} \mathbb{E}|f(r)|^2 dr dt. \qquad (3.2)
\end{aligned}
$$

Then we finish the proof if one lets ε_2 tend to 0 in (3.2).

After above preparation, we give our main result now.

Theorem 3.1 ***(Uniqueness of Stochastic Entropy Solutions)*** *Let A, B and $B_{i,j}$ be described as in (1.3), that*

$$
[\partial_v A]_+ \in L^1_{loc}([0,\infty); L^\infty(\mathbb{R}^{d+1})), \qquad (3.3)
$$

$$
\frac{A}{1+|v|} \in L^1_{loc}([0,\infty); L^1(\mathbb{R}^{d+1}) + L^\infty(\mathbb{R}^{d+1}) + L^1(\mathbb{R}^d_x; L^\infty(\mathbb{R}_v))). \qquad (3.4)
$$

Further, if one presumes that

$$
B \in L^1_{loc}([0,\infty); W^{1,\infty}_{loc}(\mathbb{R}^d;\mathbb{R}^d)), \quad B_{i,j} \in L^2_{loc}([0,\infty); W^{1,\infty}_{loc}(\mathbb{R})), \qquad (3.5)
$$

then the stochastic entropy solutions of (1.1), (1.2) is unique.

Proof. Let ρ_1 and ρ_2 be two stochastic entropy solutions of (1.1), with initial values $\rho_{0,1}$ and $\rho_{0,2}$, respectively. Then $u_1 = \chi_{\rho_1}$ and $u_2 = \chi_{\rho_2}$ defined by (1.7) are stochastic weak solutions of (1.8) with nonhomogeneous terms $\partial_v m_1$ and $\partial_v m_2$, initial data $u_{0,1} = \chi_{\rho_{0,1}}$ and $u_{0,2} = \chi_{\rho_{0,2}}$, respectively.

Let ϱ_3 be another regularization kernel in variables v, i.e.

$$
\varrho_3 \in \mathcal{D}_+(\mathbb{R}), \ \int_{\mathbb{R}} \varrho_3(v)dv = 1.
$$

For $\varepsilon_1, \varepsilon_2, \epsilon > 0$, set

$$
\varrho_{1,\varepsilon_1}(x) = \frac{1}{\varepsilon_1^d}\varrho_1(\frac{x}{\varepsilon_1}), \quad \varrho_{2,\varepsilon_2}(t) = \frac{1}{\varepsilon_2}\varrho_2(\frac{t}{\varepsilon_2}), \quad \varrho_{3,\epsilon}(v) = \frac{1}{\epsilon}\varrho_3(\frac{v}{\epsilon}),
$$

then $u_\iota^{\varepsilon,\epsilon} := u_\iota * \varrho_{1,\varepsilon_1} * \varrho_{2,\varepsilon_2} * \varrho_{3,\epsilon}$ ($\iota = 1,2$) meets

$$
\left\{ \begin{array}{l} \partial_t u_\iota^{\varepsilon,\epsilon} + b(v)\cdot\nabla_x u_\iota^{\varepsilon,\epsilon} + A(t,x,v)\partial_v u_\iota^{\varepsilon,\epsilon} + \partial_{x_i} u_\iota^{\varepsilon,\epsilon} \circ \dot{M}_i(t,v) = \partial_v m_\iota^{\varepsilon,\epsilon} + R_\iota^{\varepsilon,\epsilon}, \\ u_\iota^{\varepsilon,\epsilon}(t,x,v)|_{t=0} = \chi_{\rho_0^\iota} * \varrho_{1,\varepsilon_1} * \varrho_{3,\epsilon}(x,v), \end{array} \right. \qquad (3.6)
$$

here $R_\iota^{\varepsilon,\epsilon}=R_{\iota,1}^{\varepsilon,\epsilon}+R_{\iota,2}^{\varepsilon,\epsilon}+R_{\iota,3}^{\varepsilon,\epsilon}$, and

$$\left\{\begin{array}{l} R_{\iota,1}^{\varepsilon,\epsilon}=b(v)\cdot\nabla_x u_\iota^{\varepsilon,\epsilon}-[b(v)\cdot\nabla_x u_\iota]^{\varepsilon,\epsilon},\\ R_{\iota,2}^{\varepsilon,\epsilon}=A(t,x,v)\partial_v u_\iota^{\varepsilon,\epsilon}-[A(t,x,v)\partial_v u_\iota]^{\varepsilon,\epsilon},\\ R_{\iota,3}^{\varepsilon,\epsilon}=\partial_{x_i}u_\iota^{\varepsilon,\epsilon}\circ\dot{M}_i(t,v)-[\partial_{x_i}u_\iota\circ\dot{M}_i(t,v)]^{\varepsilon,\epsilon}.\end{array}\right. \tag{3.7}$$

From (3.6), for $\iota=1,2$, it follows that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^{d+1}}|u_\iota^{\varepsilon,\epsilon}|\xi_k(v)\theta_n(x)dxdv\\
=\;&\int_{\mathbb{R}^{d+1}}|u_\iota^{\varepsilon,\epsilon}|\xi_k(v)b(v)\cdot\nabla_x\theta_n(x)dxdv+\int_{\mathbb{R}^{d+1}}|u_\iota^{\varepsilon,\epsilon}|\partial_v[\xi_k(v)A(t,x,v)]\theta_n(x)dxdv\\
&+\int_{\mathbb{R}^{d+1}}|u_\iota^{\varepsilon,\epsilon}|\partial_{x_i}\theta_n(x)\xi_k(v)\circ\dot{M}_i(t,v)dxdv+\int_{\mathbb{R}^{d+1}}\mathrm{sign}u_\iota^{\varepsilon,\epsilon}\xi_k(v)\theta_n(x)R_\iota^{\varepsilon,\epsilon}(t,x,v)dxdv\\
&+\int_{\mathbb{R}^{d+1}}\xi_k(v)\mathrm{sign}u_\iota^{\varepsilon,\epsilon}\theta_n(x)\partial_v m_\iota^{\varepsilon,\epsilon}(t,x,v)dxdv,
\end{aligned}\tag{3.8}$$

where M_i, θ_n and ξ_k are given by (1.10), (2.15) and (2.18) respectively.

An analogue calculation also yields that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^{d+1}}u_1^{\varepsilon,\epsilon}(t,x,v)u_2^{\varepsilon,\epsilon}(t,x,v)\xi_k(v)\theta_n(x)dxdv\\
=\;&\int_{\mathbb{R}^{d+1}}u_1^{\varepsilon,\epsilon}u_2^{\varepsilon,\epsilon}\xi_k(v)b(v)\cdot\nabla_x\theta_n(x)dxdv+\int_{\mathbb{R}^{d+1}}u_1^{\varepsilon,\epsilon}u_2^{\varepsilon,\epsilon}\partial_v[\xi_k(v)A(t,x,v)]\theta_n(x)dxdv\\
&+\int_{\mathbb{R}^{d+1}}u_1^{\varepsilon,\epsilon}u_2^{\varepsilon,\epsilon}\partial_{x_i}\theta_n\xi_k\circ\dot{M}_i(t,v)dxdv+\int_{\mathbb{R}^{d+1}}\xi_k\theta_n[u_1^{\varepsilon,\epsilon}\partial_v m_2^{\varepsilon,\epsilon}+u_2^{\varepsilon,\epsilon}\partial_v m_1^{\varepsilon,\epsilon}]dxdv\\
&+\int_{\mathbb{R}^{d+1}}\xi_k(v)\theta_n(x)[R_1^{\varepsilon,\epsilon}(t,x,v)u_2^{\varepsilon,\epsilon}+R_2^{\varepsilon,\epsilon}(t,x,v)u_1^{\varepsilon,\epsilon}]dxdv.
\end{aligned}\tag{3.9}$$

From (3.8) and (3.9), one concludes

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^{d+1}}|u_1^{\varepsilon,\epsilon}|\xi_k(v)\theta_n(x)dxdv+\frac{d}{dt}\int_{\mathbb{R}^{d+1}}|u_2^{\varepsilon,\epsilon}|\xi_k\theta_n dxdv-2\frac{d}{dt}\int_{\mathbb{R}^{d+1}}u_1^{\varepsilon,\epsilon}u_2^{\varepsilon,\epsilon}\xi_k\theta_n dxdv\\
=\;&\int_{\mathbb{R}^{d+1}}[|u_1^{\varepsilon,\epsilon}|+|u_2^{\varepsilon,\epsilon}|-2u_1^{\varepsilon,\epsilon}u_2^{\varepsilon,\epsilon}]\xi_k(v)b(v)\cdot\nabla_x\theta_n(x)dxdv\\
&+\int_{\mathbb{R}^{d+1}}[|u_1^{\varepsilon,\epsilon}|+|u_2^{\varepsilon,\epsilon}|-2u_1^{\varepsilon,\epsilon}u_2^{\varepsilon,\epsilon}]\partial_v[\xi_k A(t,x,v)]\theta_n dxdv\\
&+\int_{\mathbb{R}^{d+1}}\xi_k\theta_n[\mathrm{sign}u_1^{\varepsilon,\epsilon}R_1^{\varepsilon,\epsilon}+\mathrm{sign}u_2^{\varepsilon,\epsilon}R_2^{\varepsilon,\epsilon}]dxdv-2\int_{\mathbb{R}^{d+1}}\xi_k\theta_n[R_1^{\varepsilon,\epsilon}u_2^{\varepsilon,\epsilon}+R_2^{\varepsilon,\epsilon}u_1^{\varepsilon,\epsilon}]dxdv\\
&+\int_{\mathbb{R}^{d+1}}[|u_1^{\varepsilon,\epsilon}|+|u_2^{\varepsilon,\epsilon}|-2u_1^{\varepsilon,\epsilon}u_2^{\varepsilon,\epsilon}]\partial_{x_i}\theta_n(x)\xi_k(v)\circ\dot{M}_i(t,v)dxdv+I,
\end{aligned}\tag{3.10}$$

where

$$\begin{aligned}
I\;=\;&\int_{\mathbb{R}^{d+1}}\xi_k\theta_n[\mathrm{sign}u_1^{\varepsilon,\epsilon}\partial_v m_1^{\varepsilon,\epsilon}+\mathrm{sign}u_2^{\varepsilon,\epsilon}\partial_v m_2^{\varepsilon,\epsilon}]dxdv\\
&-2\int_{\mathbb{R}^{d+1}}\xi_k\theta_n[u_1^{\varepsilon,\epsilon}\partial_v m_2^{\varepsilon,\epsilon}+u_2^{\varepsilon,\epsilon}\partial_v m_1^{\varepsilon,\epsilon}]dxdv
\end{aligned}$$

$$=: \quad I_1 - 2I_2.$$

Observing that for any $T>0$, and almost all $\omega\in\Omega$, m_1 and m_2 are bounded on $[0,T]\times\mathbb{R}^d\times\mathbb{R}$, supported in $[0,T]\times\mathbb{R}^d\times[-K,K]$ ($K=\|\rho_1\|_{L^\infty(\Omega\times(0,T)\times\mathbb{R}^d)}\vee\|\rho_2\|_{L^\infty(\Omega\times(0,T)\times\mathbb{R}^d)}$), thus for k sufficiently large,

$$\begin{aligned}
&\int_{\mathbb{R}^{d+1}}\partial_v\xi_k\theta_n[\mathrm{sign}u_1^{\varepsilon,\epsilon}m_1^{\varepsilon,\epsilon}+\mathrm{sign}u_2^{\varepsilon,\epsilon}m_2^{\varepsilon,\epsilon}]dxdv=0,\\
&\int_{\mathbb{R}^{d+1}}\partial_v\xi_k\theta_n[u_1^{\varepsilon,\epsilon}m_2^{\varepsilon,\epsilon}+u_2^{\varepsilon,\epsilon}m_1^{\varepsilon,\epsilon}]dxdv=0.
\end{aligned}\tag{3.11}$$

From $(3.6)_1$ and (1.7), with the aid of assumption (1.3) and Lemma 2.2, m_ι^ε ($\iota=1,2$) is continuous in v in a neighborhood of zero. Besides, note that

$$\mathrm{sign}u_\iota^{\varepsilon,\epsilon}\longrightarrow\mathrm{sign}u_\iota^\varepsilon=\mathrm{sign}v,\quad\text{as}\quad\epsilon\to0,$$

therefore for large k,

$$\lim_{\epsilon\to0}I_1=-2\int_{\mathbb{R}^d}\theta_n(x)[m_1^\varepsilon(t,x,0)+m_2^\varepsilon(t,x,0)]dx\tag{3.12}$$

Moreover, due to (2.12) and the fact $m_\iota\geqslant0$ ($\iota=1,2$), so for k large enough,

$$\begin{aligned}
I_2 \;=\;& \int_{\mathbb{R}^{d+1}}\varrho_{1,\varepsilon_1}(y)\varrho_{2,\varepsilon_2}(s)dyds\int_{\mathbb{R}^{d+2}}[\xi_k(\rho_1(t-s,x-y)+\tau)m_2^{\varepsilon,\epsilon}(t,x,\rho_1(t-s,x-y)+\tau)\\
&+\xi_k(\rho_2(t-s,x-y)+\tau)m_1^{\varepsilon,\epsilon}(t,x,\rho_2(t-s,x-y)+\tau)]\varrho_{3,\epsilon}(\tau)\theta_n(x)dxdvd\tau\\
&-\int_{\mathbb{R}^{d+1}}\xi_k(v)\theta_n(x)[m_1^{\varepsilon,\epsilon}(t,x,v)+m_2^{\varepsilon,\epsilon}(t,x,v)]\varrho_{3,\epsilon}(v)dxdv\\
\geqslant\;& -\int_{\mathbb{R}^{d+1}}\xi_k(v)\theta_n(x)[m_1^{\varepsilon,\epsilon}(t,x,v)+m_2^{\varepsilon,\epsilon}(t,x,v)]\varrho_{3,\epsilon}(v)dxdv\\
\longrightarrow\;& -\int_{\mathbb{R}^d}\theta_n(x)[m_1^\varepsilon(t,x,0)+m_2^\varepsilon(t,x,0)]dx,\quad\text{as}\quad\epsilon\to0,
\end{aligned}\tag{3.13}$$

On account of (3.7), thanks to Lemma 3.1, then,

$$\lim_{\varepsilon_1\to0}\lim_{\varepsilon_2\to0}\lim_{\epsilon\to0}R_{\iota,i}^{\varepsilon,\epsilon}=0,\quad\text{in}\quad L^1(\Omega;L^1(0,T;L^1_{loc}(\mathbb{R}^{d+1}))),\quad\text{for}\quad\iota,i=1,2.\tag{3.14}$$

On the other hand, for fixed ε_1, we have

$$R_{\iota,3}^{\varepsilon,\epsilon}=\partial_{x_i}u_\iota^{\varepsilon,\epsilon}\circ\dot M_i(t,v)-[\partial_{x_i}u_\iota\circ\dot M_i(t,v)]^{\varepsilon,\epsilon}:=I_{\iota,3}^{\varepsilon,\epsilon}-\frac12J_{\iota,3}^{\varepsilon,\epsilon}$$

where

$$I_{\iota,3}^{\varepsilon,\epsilon}=\partial_{x_i}u_\iota^{\varepsilon,\epsilon}\dot M_i(t,v)-[\partial_{x_i}u_\iota^{\varepsilon_1}\dot M_i(t,v)]^{\varepsilon_2,\epsilon}$$

$$J_{\iota,3}^{\varepsilon,\epsilon}=\partial^2_{x_i,x_j}u_\iota^{\varepsilon,\epsilon}\sigma_{i,k}\sigma_{j,k}(t,v)-[\partial^2_{x_i,x_j}u_\iota^{\varepsilon_1}\sigma_{i,k}\sigma_{j,k}]^{\varepsilon_2,\epsilon}$$

Due to Lemma 3.2 and (3.5),

$$\lim_{\varepsilon_1\to 0}\lim_{\varepsilon_2\to 0}\lim_{\epsilon\to 0} I^{\varepsilon,\epsilon}_{\iota,3}=0,\quad \text{in}\quad L^2(\Omega;L^2(0,T;L^2_{loc}(\mathbb{R}^{d+1}))),\tag{3.15}$$

and by virtue of Lemma 3.1,

$$\lim_{\varepsilon_1\to 0}\lim_{\varepsilon_2\to 0}\lim_{\epsilon\to 0} J^{\varepsilon,\epsilon}_{\iota,3}=0,\quad \text{in}\quad L^1(\Omega;L^1(0,T;L^1_{loc}(\mathbb{R}^{d+1}))),\tag{3.16}$$

for $\iota=1,2$.

For n and k (k is big enough) be fixed, if one let ϵ tend to zero first, ε_2 approach to zero next, ε_1 incline to zero last, with the aid of (3.11)–(3.16), from (3.10), it leads to

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^{d+1}}|u_1|\xi_k(v)\theta_n(x)dxdv+\frac{d}{dt}\int_{\mathbb{R}^{d+1}}|u_2|\xi_k\theta_n dxdv-2\frac{d}{dt}\int_{\mathbb{R}^{d+1}}u_1u_2\xi_k\theta_n dxdv\\
\leqslant\;&\int_{\mathbb{R}^{d+1}}[|u_1|+|u_2|-2u_1u_2]\xi_k(v)b(v)\cdot\nabla_x\theta_n(x)dxdv\\
&+\int_{\mathbb{R}^{d+1}}[|u_1|+|u_2|-2u_1u_2]\partial_v[\xi_k A(t,x,v)]\theta_n dxdv\\
&+\int_{\mathbb{R}^{d+1}}[|u_1|+|u_2|-2u_1u_2]\partial_{x_i}\theta_n(x)\xi_k(v)\circ\dot{M}_i(t,v)dxdv.
\end{aligned}\tag{3.17}$$

Because of the fact $|u_1-u_2|^2=|u_1-u_2|$, from (3.17), one derives

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}\int_{\mathbb{R}^{d+1}}|u_1-u_2|\xi_k(v)\theta_n(x)dxdv\\
\leqslant\;&\mathbb{E}\int_{\mathbb{R}^{d+1}}|u_1-u_2|\xi_k(v)b(v)\cdot\nabla_x\theta_n(x)dxdv\\
&+\mathbb{E}\int_{\mathbb{R}^{d+1}}|u_1-u_2|\partial_v[\xi_k(v)A(t,x,v)]\theta_n(x)dxdv\\
&+\mathbb{E}\int_{\mathbb{R}^{d+1}}|u_1-u_2|\partial_{x_i}\theta_n(x)\xi_k(v)\circ\dot{M}_i(t,v)dxdv.
\end{aligned}\tag{3.18}$$

By taking n to infinity first, k to infinity second, with the help of (3.4), (3.5), then

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}\int_{\mathbb{R}^{d+1}}|u_1-u_2|dxdv\;&\leqslant\;\mathbb{E}\int_{\mathbb{R}^{d+1}}|u_1-u_2|\partial_vA(t,x,v)dxdv\\
&\leqslant\;\|[\partial_vA(t,\cdot,\cdot)]^+\|_{L^\infty(\mathbb{R}^{d+1})}\mathbb{E}\int_{\mathbb{R}^{d+1}}|u_1-u_2|dxdv,
\end{aligned}$$

where in the last inequality, we have used the assumption (3.3).

Thus

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^d}|\rho_1(t)-\rho_2(t)|dx\\
=\;&\mathbb{E}\int_{\mathbb{R}^{d+1}}|u_1(t)-u_2(t)|dxdv\\
\leqslant\;&\int_{\mathbb{R}^{d+1}}|u_{0,1}-u_{0,2}|dxdv\exp(\int_0^t\|[\partial_vA(s,\cdot,\cdot)]^+\|_{L^\infty(\mathbb{R}^{d+1})}ds)
\end{aligned}$$

$$= \int_{\mathbb{R}^d} |\rho_{0,1}(x) - \rho_{0,2}(x)| dx \exp(\int_0^t \|[\partial_v A(s,\cdot,\cdot)]^+\|_{L^\infty(\mathbb{R}^{d+1})} ds) \tag{3.19}$$

From (3.19), we complete the proof.

From Theorem 3.1, one clearly has the below comparison result.

Corollary 3.1 ***(Comparison Principle)*** *Let ρ_1 and ρ_2 be two stochastic entropy solutions of (1.1), with initial values $\rho_{0,1}$ and $\rho_{0,2}$, if $\rho_{0,1} \leqslant \rho_{0,2}$, then with probability 1, $\rho_1 \leqslant \rho_2$.*

Proof. Clearly, mimicking above calculation, we have

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}\int_{\mathbb{R}^{d+1}} [u_1(t,x,v) - u_2(t,x,v)] dxdv \\
= \ & \mathbb{E}\int_{\mathbb{R}^{d+1}} [u_1(t,x,v) - u_2(t,x,v)] \partial_v A(t,x,v) dxdv.
\end{aligned}$$

Observing that

$$[u_1(t,x,v) - u_2(t,x,v)]^- = \frac{|u_1 - u_2| - (u_1 - u_2)}{2},$$

hence

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}\int_{\mathbb{R}^{d+1}} [u_1(t,x,v) - u_2(t,x,v)]^- dxdv \\
= \ & \frac{1}{2}\mathbb{E}\frac{d}{dt}\int_{\mathbb{R}^{d+1}} |u_1(t,x,v) - u_2(t,x,v)| - \frac{1}{2}\mathbb{E}\frac{d}{dt}\int_{\mathbb{R}^{d+1}} [u_1(t,x,v) - u_2(t,x,v)] \\
\leqslant \ & \frac{1}{2}\mathbb{E}\int_{\mathbb{R}^{d+1}} \Big[|u_1(t,x,v) - u_2(t,x,v)| - u_1(t,x,v) + u_2(t,x,v)\Big] \partial_v A(t,x,v) dxdv \\
= \ & \mathbb{E}\int_{\mathbb{R}^{d+1}} [u_1(t,x,v) - u_2(t,x,v)]^- \partial_v A(t,x,v) dxdv \\
\leqslant \ & \mathbb{E}\int_{\mathbb{R}^{d+1}} [u_1(t,x,v) - u_2(t,x,v)]^- dxdv \|[\partial_v A(t,\cdot,\cdot)]^+\|_{L^\infty(\mathbb{R}^{d+1})}.
\end{aligned}$$

Then the Grönwall inequality applies, one concludes

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^d} [\rho_1(t,x) - \rho_2(t,x)]^- dx \\
= \ & \mathbb{E}\int_{\mathbb{R}^{d+1}} [u_1(t,x,v) - u_2(t,x,v)]^- \partial_v A(t,x,v) dxdv \\
\leqslant \ & \int_{\mathbb{R}^{d+1}} [u_{0,1}(x,v) - u_{0,2}(x,v)]^- dxdv \exp(\int_0^t \|[\partial_v A(s,\cdot,\cdot)]^+\|_{L^\infty(\mathbb{R}^{d+1})} ds) \\
= \ & \int_{\mathbb{R}^{d+1}} [\rho_{0,1}(x) - \rho_{0,2}(x)]^- dx \exp(\int_0^t \|[\partial_v A(s,\cdot,\cdot)]^+\|_{L^\infty(\mathbb{R}^{d+1})} ds) \\
= \ & 0,
\end{aligned}$$

which implies $\rho_1 \leqslant \rho_2, \quad P - a.s.$

Remark 3.1 *(i) As a special case, one confirms the uniqueness of stochastic entropy solutions for*

$$\begin{cases} \partial_t \rho(t,x) + div_x(B(\rho)) + \nabla\rho(t,x)\cdot\circ\dot{W}(t) = A(t,x,\rho), \quad in \quad \Omega\times(0,\infty)\times\mathbb{R}^d, \\ \rho(t,x)|_{t=0} = \rho_0(x) \quad in \quad \Omega\times\mathbb{R}^d, \end{cases}$$

when $B \in W^{1,\infty}_{loc}(\mathbb{R};\mathbb{R}^d)$. However, we can not give an affirm answer on the problem whether the weak solutions is unique or not, when B is non-regular (such as $B \in L^\infty(\mathbb{R};\mathbb{R}^d)$).

(ii) Our proof originates from [13], but the calculation here seems to be finer and the demonstration is more difficult, for more details one can pay his attention on [13] and the references cited up there.

To make our discussion on uniqueness more clear, we exhibit a representative example here.

Example 3.1 *A porous medium equation (see [14]) with a nonlinear source, a nonlinear convection term and a stochastic perturbation reads*

$$\begin{cases} \partial_t \rho + div_x(\zeta|\rho|^\alpha\rho) + \vartheta(t)\partial_{x_i}(|\rho|^{\beta/2}\rho)\circ\dot{W}_i(t) = A(t,\rho), \quad in \quad \Omega\times(0,\infty)\times\mathbb{R}^d, \\ \rho(t,x)|_{t=0} = \rho_0(x), \quad in \quad \Omega\times\mathbb{R}^d, \end{cases} \tag{3.20}$$

where $\zeta \in \mathbb{R}^d$ is a fixed vector, $\alpha, \beta \geqslant 0$ are constants,

$$\vartheta(t) \in L^2_{loc}([0,\infty)), \quad A(t,\rho) = \frac{\lambda(t)\rho^2}{1+\rho^2}, \quad \lambda(t) \in L^1_{loc}([0,\infty)).$$

From Theorem 3.1 and Corollary 3.1, we have

Proposition 3.1 *Let $\rho_0 \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, then the stochastic entropy solutions of (3.20) is unique. In addition, if $\rho_0 \geqslant 0$, the unique stochastic entropy solution $\rho \geqslant 0$.*

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